

Tetravalent 2-transitive Cayley graphs of finite simple groups and their automorphism groups*

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Abstract

A graph Γ is called (G, s) -arc-transitive if $G \leq \text{Aut}(\Gamma)$ is transitive on $V\Gamma$ and transitive on the set of s -arcs of Γ , where for an integer $s \geq 1$ an s -arc of Γ is a sequence of $s + 1$ vertices (v_0, v_1, \dots, v_s) of Γ such that v_{i-1} and v_i are adjacent for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$. Γ is called 2-transitive if it is $(\text{Aut}(\Gamma), 2)$ -arc-transitive but not $(\text{Aut}(\Gamma), 3)$ -arc-transitive. A Cayley graph Γ of a group G is called normal if G is normal in $\text{Aut}(\Gamma)$ and non-normal otherwise. It was proved by X. G. Fang, C. H. Li and M. Y. Xu that if Γ is a tetravalent 2-transitive Cayley graph of a finite simple group G , then either Γ is normal or G is one of the groups $\text{PSL}_2(11)$, M_{11} , M_{23} and A_{11} . In the present paper we prove further that among these four groups only M_{11} produces connected tetravalent 2-transitive non-normal Cayley graphs, and there are exactly two such graphs which are non-isomorphic and both determined in the paper. As a consequence, the automorphism group of any connected tetravalent 2-transitive Cayley graph of any finite simple group is determined.

Keywords: Cayley graph; s -arc-transitive graph; 2-transitive graph; finite simple group

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1 Introduction

All groups considered in the paper are finite, and all graphs considered are finite, simple and undirected. Given a group G and a subset S of G such that $1_G \notin S$ and $S = S^{-1} := \{x^{-1} : x \in S\}$, the *Cayley graph* of G relative to S is defined to be the graph $\Gamma = \text{Cay}(G, S)$ with vertex set $V\Gamma = G$ and edge set $E\Gamma = \{\{x, y\} \mid yx^{-1} \in S\}$. It is readily seen that Γ is connected if and only if S is a generating set of G . In general, Γ has exactly $|G : \langle S \rangle|$ connected components, each of which is isomorphic to $\text{Cay}(\langle S \rangle, S)$, where $\langle S \rangle$ is the subgroup of G generated by S . So we may focus on the connected case when dealing with Cayley graphs. Denote by G_R the right regular representation of G . Define

$$A(G, S) := \{x \in \text{Aut}(G) \mid S^x = S\}.$$

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Then $A(G, S)$ is a subgroup of $\text{Aut}(G)$ acting naturally on G . It is not difficult to see that $\Gamma = \text{Cay}(G, S)$ admits $G_RA(G, S)$ as a subgroup of its full automorphism group $\text{Aut}(\Gamma)$. It is well known (see [6, 12]) that $N_{\text{Aut}(\Gamma)}(G_R) = G_RA(G, S)$. Since $G_R \cong G$, we may use G in place of G_R , so that $G_RA(G, S)$ is written as $G.A(G, S)$. Γ is called a *normal* Cayley graph if G is normal in $\text{Aut}(\Gamma)$, that is, $\text{Aut}(\Gamma) = G.A(G, S)$.

A fundamental problem in studying the structure of a graph is to determine its full automorphism group. This is, in general, quite difficult. However, for a connected Cayley graph $\Gamma = \text{Cay}(G, S)$ of valency d , if Γ is normal, then we know that its automorphism group is given by $\text{Aut}(\Gamma) = G.A(G, S)$. Moreover, the subgroup $A(G, S)$ of $\text{Aut}(G)$ acts faithfully on the *neighbourhood* $\Gamma(\alpha)$ of $\alpha \in V\Gamma$, where $\Gamma(\alpha)$ is defined as the set of vertices of Γ adjacent to α in Γ . Hence $A(G, S)$ is isomorphic to a subgroup of the symmetric group S_d of degree d . In other words, if Γ is a normal Cayley graph, then the structure of $\text{Aut}(\Gamma)$ is well understood. In contrast, it is more challenging to determine the automorphism groups of non-normal Cayley graphs. As such non-normal Cayley graphs have attracted considerable attention in recent years.

Given an integer $s \geq 0$, an s -arc of a graph Γ is a sequence (v_0, v_1, \dots, v_s) of $s + 1$ vertices of Γ such that $\{v_{i-1}, v_i\} \in E\Gamma$ for $i = 1, 2, \dots, s$ and $v_{i-1} \neq v_{i+1}$ for $i = 1, 2, \dots, s - 1$. A graph Γ is called (G, s) -arc-transitive if G is a subgroup of $\text{Aut}(\Gamma)$ that is transitive on $V\Gamma$ and transitive on the set of s -arcs of Γ . A (G, s) -arc-transitive graph is called (G, s) -transitive if it is not $(G, s + 1)$ -arc-transitive. In particular, Γ is called s -arc-transitive if it is $(\text{Aut}(\Gamma), s)$ -arc-transitive, and s -transitive if it is $(\text{Aut}(\Gamma), s)$ -transitive. A 1-arc-transitive graph is also called an *arc-transitive* or *symmetric* graph.

For any integer $s \geq 1$, a complete classification of cubic s -transitive non-normal Cayley graphs of finite simple groups was obtained by S. J. Xu, M. Y. Xu and the first two authors of the present paper (see [13, 14]). In the tetravalent case, C. H. Li, M. Y. Xu and the first author of the present paper proved ([2, Theorem 1.1]) that, if Γ is a tetravalent 2-transitive Cayley graph of a finite simple group G , then either Γ is normal or G is one of the groups $\text{PSL}_2(11)$, M_{11} , M_{23} and A_{11} . It was unknown whether Γ is normal when G is one of these four groups. In this paper we settle these unsolved cases and classify all connected tetravalent 2-transitive non-normal Cayley graphs of finite simple groups. As a consequence, the automorphism group of any connected tetravalent 2-transitive Cayley graph of any finite simple group is determined.

The main result of this paper is as follows, where the graphs $\Gamma(\Delta_1)$ and $\Gamma(\Delta_2)$ involved will be defined in (3) in Section 3.

Theorem 1.1. *Let G be a finite nonabelian simple group and $\Gamma = \text{Cay}(G, S)$ a connected tetravalent 2-transitive Cayley graph of G . Then one of the following occurs:*

- (a) Γ is normal, and $\text{Aut}(\Gamma) = G.A_4$ or $G.S_4$;
- (b) $G = M_{11}$, $\text{Aut}(\Gamma) = \text{Aut}(M_{12}) = M_{11}S_4$, $\text{Aut}(\Gamma)_\alpha \cong S_4$ for $\alpha \in V\Gamma$, Γ is non-normal, $\Gamma \cong \Gamma(\Delta_1)$ or $\Gamma(\Delta_2)$, and $\Gamma(\Delta_1)$ and $\Gamma(\Delta_2)$ are not isomorphic.

In the next section we give some notations and preliminary results. In Section 3 we determine all tetravalent 2-transitive non-normal Cayley graphs of finite simple groups by analyzing the four groups above. In Section 4 we settle the isomorphism problem and thus complete the proof of Theorem 1.1. As we will see, even in the four innocent-looking cases above, considerable analysis and computation will be needed in order to establish Theorem 1.1.

2 Preliminaries

A permutation group G acting on a set Ω is said to be *quasiprimitive* if each of its nontrivial normal subgroups is transitive on Ω . The *socle* of a group G , denoted by $\text{soc}(G)$, is the product of all minimal normal subgroups of G . In particular, G is said to be *almost simple* if $\text{soc}(G)$ is a nonabelian simple group. Given a graph Γ and a group $K \leq \text{Aut}(\Gamma)$, the *quotient graph* Γ_K of Γ relative to K is defined as the graph with vertices the K -orbits on $V\Gamma$, such that two K -orbits, say X and Y , are adjacent in Γ_K if and only if there is an edge of Γ with one end-vertex in X and the other end-vertex in Y .

The following lemma determines the vertex stabilizers for connected tetravalent 2-transitive graphs (see [10, Theorem 4] or [7, Proposition 2.2]).

Lemma 2.1. *Let Γ be a connected tetravalent 2-transitive graph. Then the vertex stabilizer of Γ is A_4 or S_4 .*

The next lemma describes possible structure of the full automorphism group of a connected Cayley graph of a finite simple group.

Lemma 2.2. ([4, Theorem 1.1]) *Let G be a finite nonabelian simple group and $\Gamma = \text{Cay}(G, S)$ a connected Cayley graph of G . Let M be a subgroup of $\text{Aut}(\Gamma)$ containing $G.A(G, S)$. Then either $M = G.A(G, S)$ or one of the following holds:*

- (a) *M is almost simple, and $\text{soc}(M)$ contains G as a proper subgroup and is transitive on $V\Gamma$;*
- (b) *$G \cdot \text{Inn}(G) \leq M = G \cdot A(G, S) \cdot 2$ and S is a self-inverse union of G -conjugacy classes;*
- (c) *M is not quasiprimitive and there is a maximal intransitive normal subgroup H of M such that one of the following holds:*
 - (i) *M/H is almost simple, and $\text{soc}(M/H)$ contains $GH/H \cong G$ and is transitive on $V\Gamma_H$;*
 - (ii) *$M/H = \text{AGL}_3(2)$, $G = \text{L}_2(7)$, and $\Gamma_H \cong K_8$;*
 - (iii) *$\text{soc}(M/H) \cong T \times T$, and $GH/H \cong G$ is a diagonal subgroup of $\text{soc}(M/H)$, where T and G are given in Table 1.*

Moreover, there are examples of connected Cayley graphs of finite simple groups in each of these cases.

A subgroup K of a group G is called *core-free* if $\bigcap_{g \in G} K^g = 1$. Given a core-free subgroup K of G and an element $g \in G \setminus N_G(K)$ such that $g^2 \in K$ and $G = \langle K, g \rangle$, the *coset graph* $\Gamma^* = \Gamma(G, K, g)$ is defined by

$$V\Gamma^* = [G : K] = \{ Kx \mid x \in G \}, \quad E\Gamma^* = \{ \{Kx, Ky\} \mid xy^{-1} \in KgK \}.$$

A well known result due to Sabidussi [11] and Lorimer [9] asserts that Γ^* is G -arc-transitive and up to isomorphism every G -arc-transitive graph can be constructed this way. The following lemma is a refinement of this result (see [3, Theorem 2.1]).

Lemma 2.3. *Let Γ be a finite connected $(G, 2)$ -arc-transitive graph of valency d . Then there exists a core-free subgroup K of G and an element $g \in G$ such that*

	G	T	$ V\Gamma_K $
1	A_6	G	36
2	M_{12}	G or A_m	144
3	$\text{Sp}_4(q)$ ($q = 2^a > 2$)	G or A_m or $\text{Sp}_{4r}(q_0)$ ($q = q_0^r$)	$\frac{q^4(q^2-1)^2}{4}$
4		$\text{Sp}_{4r}(q_0)$ ($q = q_0^r$)	$\frac{q^4(q^2-1)^2}{2}$
5	$\text{P}\Omega_8^+(q)$	G or A_m or $\text{Sp}_8(2)$ (if $q = 2$)	$\frac{q^6(q^4-1)^2}{(2,q-1)^2}$

Table 1: Lemma 2.2 (c)(iii).

- (a) $g \notin N_G(K)$, $g^2 \in G$, $\langle K, g \rangle = G$;
- (b) the action of K on $[K : K \cap K^g]$ by right multiplication is transitive, where $|K : K \cap K^g| = d$; and
- (c) $\Gamma \cong \Gamma(G, K, g)$.

Moreover, one can choose g to be a 2-element.

Conversely, if G is a finite group with a core-free subgroup K and an element g satisfying (a) and (b) above, then $\Gamma^* = \Gamma(G, K, g)$ is a connected $(G, 2)$ -arc-transitive graph, and G acts faithfully on the vertex set $[G : K]$ of Γ^* by right multiplication.

3 Tetravalent 2-transitive non-normal Cayley graphs

The purpose of this section is to prove the following proposition, which gives all tetravalent 2-transitive non-normal Cayley graphs of finite simple groups. We postpone the definition of $\Gamma(\Delta_1)$ and $\Gamma(\Delta_2)$ to (3).

Proposition 3.1. *Let G be a finite simple group and Γ a connected tetravalent 2-transitive non-normal Cayley graph of G . Then $G = M_{11}$, $\text{Aut}\Gamma = \text{Aut}(M_{12}) = M_{12}:2$, $(\text{Aut}\Gamma)_\alpha = S_4$, and Γ is isomorphic to $\Gamma(\Delta_1)$ or $\Gamma(\Delta_2)$.*

Proof. Suppose that $\Gamma = \text{Cay}(G, S)$ is a connected tetravalent 2-transitive non-normal Cayley graph. Then, by [2, Theorem 1.1], G is one of the following groups:

$$\text{PSL}_2(11), M_{11}, M_{23}, A_{11}. \quad (1)$$

Write $A = \text{Aut}\Gamma$. Then $A = GA_\alpha$ with $G \cap A_\alpha = 1$ and $A_\alpha = A_4$ or S_4 by Lemma 2.1. We consider the following two situations separately.

Situation 1: A is quasiprimitive on $V\Gamma$.

In this situation, by Lemma 2.2 we know that A is almost simple. Let $T = \text{soc}(A)$. Note that $|S_4| = 24$ is divisible by $|A : G|$. It follows that (T, G) is one of the following pairs:

$$(M_{11}, \text{PSL}_2(11)), (M_{12}, M_{11}), (M_{24}, M_{23}), (A_{12}, A_{11}). \quad (2)$$

Case 1: $(T, G) \in \{(M_{11}, \text{PSL}_2(11)), (M_{24}, M_{23}), (A_{12}, A_{11})\}$

First we consider the case $(T, G) = (M_{11}, \text{PSL}_2(11))$ and suppose $M_{11} = \text{PSL}_2(11)A_4$. It is well known that M_{11} has a faithful permutation representation of degree 12 acting on $\Omega = \{1, 2, \dots, 12\}$. In this representation, $\text{PSL}_2(11)$ is the point-stabilizer and the subgroup A_4 should be regular on Ω . However, according to the permutation character $\chi = \chi_1 + \chi_{11}$ taken from ATLAS [1, p. 18], we have $\chi(1A) = 12$, $\chi(2A) = 4$ and $\chi(3A) = 3$. Therefore, the number of orbits of A_4 on Ω is

$$\frac{1}{|A_4|} \sum_{g \in A_4} \chi(g) = \frac{1}{12}(12 \cdot 1 + 4 \cdot 3 + 3 \cdot 8) = 4,$$

which contradicts the regularity of A_4 .

Next assume $(T, G) = (M_{24}, M_{23})$. In this case $M_{24} = M_{23}K$ for some subgroup $K \cong S_4$. Since K is regular on $\Omega = \{1, 2, \dots, 24\}$, following the notation of [1, p. 96], the involution of K must be in class $2B$ and the elements of order 3 in $3B$. There are two classes of regular elements of order 4, namely $4A$ and $4C$. However, the power map shows that $4A^2 = 2A$, which can not be the case. So the elements of order 4 in K must be in $4C$. Now suppose that a 2-element $g \in M_{24}$ satisfies (a) and (b) in Lemma 2.3. Since $8A^2 = 4B$, $4A^2 = 2A$ and $4B^2 = 2A$, we can only have $g \in 2A$, $2B$ or $4C$. However, an exhaustive search shows that, for such an element g , the subgroup $\langle K, g \rangle \not\cong M_{24}$, a contradiction.

Finally we consider $(T, G) = (A_{12}, A_{11})$. If $A = A_{12}$, then $A = A_{11}K$ for some subgroup $K \cong A_4$. Since K is regular on $\Omega = \{1, 2, \dots, 12\}$, the involution in K must be in conjugacy class $2B$ and the elements of order 3 in class $3C$, following the notation of [1, p. 92]. Suppose that a 2-element $g \in A_{12}$ satisfies (a) and (b) in Lemma 2.3. It is evident that g has order 2 or 4. According to the power map of conjugacy classes of A_{12} , if g has order 4, then g^2 can not be in $2B$. Thus g must have order 2. Furthermore, $|K \cap K^g| = 3$ implies that g normalizes the element of $3C$. With the help of this information, an exhaustive search shows that $\langle K, g \rangle$ can not be A . Similarly, when $A = S_{12}$, there is no 2-element satisfying (a) and (b) in Lemma 2.3.

The argument above shows that Case 1 does not occur.

Case 2: $(T, G) = (M_{12}, M_{11})$

In this case Γ is either $(\text{Aut}(M_{12}), 2)$ -arc transitive or $(M_{12}, 2)$ -arc-transitive. Consider first $\text{Aut}(M_{12}) = M_{12}:2$. This group contains a unique class of subgroups isomorphic to M_{11} . Since $|A : G| = 24$, $A_\alpha = S_4$ by Lemma 2.1. Computation using GAP yields the following:

- (a) $A = M_{12}:2$ has a unique class of subgroups $K \cong S_4$ such that $K \cap M_{11} = 1$;
- (b) for a subgroup K in (a), there are in total sixteen 2-elements $g \in A$ such that K and g satisfy (a) and (b) in Lemma 2.3; denote the set of these 16 elements by Δ ;
- (c) $N_A(K) = K \cong S_4$ and the conjugate action of K on Δ produces two orbits, denoted by Δ_1 and Δ_2 , with $|\Delta_1| = 12$ and $|\Delta_2| = 4$.

Let $K = S_4$ be a subgroup obtained in (a). For any g satisfying (b), the coset graph $\Gamma(M_{12}:2, S_4, g)$ must be a non-normal 2-transitive tetravalent Cayley graph of M_{11} . Moreover, for a coset graph $\Gamma(G, K, g)$, it is not difficult to verify that $\Gamma(G, K, g) \cong \Gamma(G, K^x, g^x)$ for any

$x \in \text{Aut}(G)$ (see [3, Fact 2.2]). It then follows that all coset graphs $\Gamma(M_{12}:2, K, g)$ with $g \in \Delta_i$ are isomorphic, for $i = 1, 2$. Fix $g_i \in \Delta_i$ for $i = 1, 2$. Define

$$\Gamma(\Delta_i) = \Gamma(M_{12}:2, S_4, g_i), \quad i = 1, 2. \quad (3)$$

These two graphs are, up to isomorphism, the only tetravalent 2-transitive non-normal Cayley graphs of M_{11} , for $\text{Aut}(\Gamma) = M_{12}:2$.

Next we consider $\Gamma(M_{12}, K, g)$. Computation shows that M_{12} has a unique class of subgroups $K \cong A_4$ satisfying $K \cap M_{11} = 1$. So we may choose $K = A_4$ such that A_4 is a subgroup of S_4 given in the previous case. In addition, there are in total twelve 2-elements g such that K and g satisfy (a) and (b) in Lemma 2.3. Moreover, these 2-elements are all in Δ_1 above and K is transitive on Δ_1 by conjugate action. Thus, up to isomorphism, we obtain a unique tetravalent 2-transitive non-normal Cayley graph of M_{11} , which is isomorphic to $\Gamma^*(\Delta_1) = \Gamma(M_{12}, A_4, g_1)$, for $g_1 \in \Delta_1$.

We claim that $\Gamma^*(\Delta_1) = \Gamma(M_{12}, A_4, g_1)$ and $\Gamma(\Delta_1) = \Gamma(M_{12}:2, S_4, g_1)$ are isomorphic. Note that A_4 is contained in S_4 and M_{12} is transitive on both $V\Gamma^*(\Delta_1)$ and $V\Gamma(\Delta_1)$. Define

$$\sigma : A_4x \mapsto S_4x, \quad x \in M_{12}.$$

It is straightforward to verify that σ is an isomorphism from $\Gamma^*(\Delta_1)$ to $\Gamma(\Delta_1)$.

Therefore, any quasiprimitive tetravalent 2-transitive non-normal Cayley graph of a finite simple group is isomorphic to $\Gamma(\Delta_1)$ or $\Gamma(\Delta_2)$.

Situation 2: A is not quasiprimitive on $V\Gamma$.

In this case, let H be a maximal intransitive normal subgroup of A . Recall that $A = GA_\alpha$ with $G \cap A_\alpha = 1$, where $A_\alpha \cong A_4$ or S_4 . By Lemma 2.2, we see that only (c)(i) in Lemma 2.2 occurs. This means that A/H is an almost simple group, and $\text{soc}(A/H)$ contains $GH/H \cong G$ and is transitive on $V\Gamma_H$, where Γ_H is the quotient graph of Γ relative to H . Set $T = \text{soc}(A/H)$.

Case 1: $T \cong G$

Since G is simple and $H \triangleleft A$, we have $H \cap G = 1$, which implies that $|H|$ is a divisor of $|S_4| = 24$. If G acts on H nontrivially by conjugation, then G is isomorphic to a subgroup of $\text{Aut}(H)$. On the other hand, it is not hard to verify that this is not the case for $G = \text{PSL}_2(11)$, M_{11} , M_{23} or A_{11} . So we assume that $GH = G \times H$. Now $T \cong G$. It follows that $|\text{Out}(T)| = 1$ for $G = M_{11}$ and $G = M_{23}$, while $|\text{Out}(T)| = 2$ for $G = \text{PSL}_2(11)$ and $G = A_{11}$. In the former case we have $A = G \times H$ and hence $G \triangleleft A$, which is impossible. In the latter case we have $|A : G \times H| = 1$ or 2 , which implies that $G \triangleleft A$, a contradiction. So Case 1 does not occur.

Case 2: $T \not\cong G$

Clearly, $G \cap H = 1$ and $|H|$ divides $|A_\alpha|$. So 24 is divisible by $|H|$. If 3 divides $|H|$, then $|T : GH/H|$ is a divisor of $8 = 2^3$, which is impossible by [8]. So H is a 2-group with $|H|$ dividing 8. Further, if $H_\alpha \neq 1$, then $d(\Gamma_H) = 2$, and hence $\text{Aut}(\Gamma_H)$ is a dihedral group, a contradiction. Hence H is semiregular on $V\Gamma$ and $d(\Gamma_H) = d(\Gamma) = 4$.

For $\alpha = 1 \in G = V\Gamma$, set $\bar{\alpha} = \alpha^H$. Since Γ is A -arc-transitive, Γ_H is A/H -arc transitive. Moreover, since $(A/H)_{\bar{\alpha}} = \{Hx \mid x \in A_\alpha\}$ and $H \cap A_\alpha = 1$, $(A/H)_{\bar{\alpha}} \cong A_\alpha$. From this it follows that Γ_H is $(A/H, 2)$ -arc transitive.

Next we determine all pairs (T, G) . Note that $|A : G|$ divides 24. So $|A/H : GH/H|$ divides $24/|H|$. Since H is a 2-group, $24/|H|$ is 6 or 12. Hence, by [1], (T, G) must be one of the following pairs:

$$(M_{11}, \text{PSL}_2(11)), (M_{12}, M_{11}), (A_{12}, A_{11}). \quad (4)$$

From this we obtain that $|H| = 2$ and $\text{soc}(A/H) = A/H = T$. Thus Γ_H is $(T, 2)$ -arc-transitive with $|V\Gamma_H| = |G|/2$ and (T, G) given in (4).

Finally, we construct all $(T, 2)$ -arc-transitive graphs for (T, G) as given in (4). Note that $|T|/|G| = 12$, $|V\Gamma_H| = |G|/2$ and $|T_{\bar{\alpha}}| = 24$. So $T_{\bar{\alpha}} \cong S_4$.

Consider $T = M_{11}$ first. There is only one class of subgroups isomorphic to S_4 . Let K be such a subgroup. Computation using GAP shows that there is no 2-element g in T satisfying (a) and (b) in Lemma 2.3, which is a contradiction.

Consider $T = M_{12}$. Computation shows that there are four classes of subgroups isomorphic to S_4 . Using GAP, we obtain that there is no 2-element $g \in T$, together with $K \cong S_4$, satisfying (a) and (b) in Lemma 2.3, which is a contradiction.

Finally, consider $T = A_{12}$. There are 24 conjugate classes of subgroups $K \cong S_4$. A systematic search using GAP shows that there is no 2-element $g \in T$ such that K and g satisfy (a) and (b) in Lemma 2.3. This completes the proof of Proposition 3.1. \square

4 Proof of Theorem 1.1

By Proposition 3.1, a connected tetravalent 2-transitive non-normal Cayley graph of a finite simple group is isomorphic to $\Gamma(\Delta_1)$ or $\Gamma(\Delta_2)$. In this section we prove that these two graphs are non-isomorphic.

Proposition 4.1. *The graphs $\Gamma(\Delta_1)$ and $\Gamma(\Delta_2)$ defined in (3) are not isomorphic.*

Proof. Write $\Gamma_i = \Gamma(\Delta_i)$ and $X_i = \text{Aut}\Gamma_i$, for $i = 1, 2$. It follows from Proposition 3.1 that Γ_1 and Γ_2 have the same vertex set and full automorphism group. Denote $V = V\Gamma_1 = V\Gamma_2$ and $X = X_1 = X_2 = \text{Aut}(M_{12}) = M_{12}:2$. Now $X_{\alpha} = S_4$. Suppose by way of contradiction that $\Gamma_1 \cong \Gamma_2$. Let ϕ be an isomorphism from Γ_1 to Γ_2 . Then $\phi \in N_{\text{Sym}(V)}(X)$ by [3, Fact 2.3]. Write $N = N_{\text{Sym}(V)}(X)$ and $C = C_{\text{Sym}(V)}(X)$. Then N/C is isomorphic to a subgroup of $\text{Aut}(X)$. Moreover, since the vertex stabilizer $X_{\alpha} \cong S_4$ is self-normalized in X (see Case 2, result (c) of Situation 1 in Section 3), $C = 1$ by [3, Proposition 2.4] and hence N is a subgroup of $\text{Aut}(X)$. Note that $X = \text{Aut}(M_{12})$ and $\text{Out}(X) = 1$. Thus $N = X$. It follows that $\phi \in X$ is an automorphism of Γ_1 , which implies that $\Gamma_1 = \Gamma_2$. On the other hand, for $\alpha = S_4 \in V$, the neighbourhood $\Gamma_i(\alpha)$ of α in Γ_i is given by

$$\Gamma_i(\alpha) = \{S_4 g_i x \mid x \in S_4\}, \text{ for } i = 1, 2.$$

However, computation shows that $\Gamma_1(\alpha) \neq \Gamma_2(\alpha)$, which contradicts the statement that $\Gamma_1 = \Gamma_2$. Therefore, Γ_1 and Γ_2 are not isomorphic. \square

Theorem 1.1 now follows from [2, Theorem 1.1], Lemma 2.1, Proposition 3.1 and Proposition 4.1 immediately.

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